

Spectral asymptotics for δ' interaction supported by a infinite curve

Michal Jex

*Doppler Institute for Mathematical Physics and Applied Mathematics,
Czech Technical University in Prague, Břehová 7, 11519 Prague,
and Department of Physics, Faculty of Nuclear Sciences and Physical Engineering, Czech
Technical University in Prague, Břehová 7, 11519 Prague, Czech republic
E-mail: jermicha@fjfi.cvut.cz

We consider a generalized Schrödinger operator in $L^2(\mathbb{R}^2)$ describing an attractive δ' interaction in a strong coupling limit. δ' interaction is characterized by a coupling parameter β and it is supported by a C^4 -smooth infinite asymptotically straight curve Γ without self-intersections. It is shown that in the strong coupling limit, $\beta \rightarrow 0_+$, the eigenvalues for a non-straight curve behave as $-\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta|\ln \beta|)$, where μ_j is the j -th eigenvalue of the Schrödinger operator on $L^2(\mathbb{R})$ with the potential $-\frac{1}{4}\gamma^2$ where γ is the signed curvature of Γ .

Keywords: δ' interaction, quantum graphs, spectral theory

1. Introduction

The quantum mechanics describing the particle confined to various manifolds is studied quite extensively. It is very useful for describing various nanostructures in physics but it also offers a large variety of interesting problems from the purely mathematical point of view. Systems where the confinement is realized by a singular attractive potential, so called 'leaky' quantum graphs [1], have the advantage that they take quantum tunneling effects into account in contrast to quantum graphs [2]. The confining potential is often taken to be of the δ type. One can think also about more singular types of potentials namely δ' type based on the concept of δ' interaction in one dimension [3].

We are interested in the spectrum of the operator which can be formally written as

$$H = -\Delta - \beta^{-1}\delta'(\cdot - \Gamma)$$

where δ' interaction is supported by an infinite curve Γ in \mathbb{R}^2 . We are interested in the strong coupling regime which corresponds to small values of the parameter β . We derive spectral asymptotics of discrete and essential spectra. As a byproduct we obtain that for a non-straight curve the bound state arises for sufficiently small β in an alternative way to one presented in [4].

2. Formulation of the Problem and Results

We consider a curve Γ parameterized by its arc length

$$\Gamma : \mathbb{R} \rightarrow \mathbb{R}^2, \quad s \mapsto (\Gamma_1(s), \Gamma_2(s)),$$

where $\Gamma_1(s), \Gamma_2(s) \in C^4(\mathbb{R})$ are component functions. We denote signed curvature as $\gamma(s) := (\Gamma_1''\Gamma_2' - \Gamma_1'\Gamma_2'')(s)$. We introduce several conditions for the curve Γ as:

- ($\Gamma 1$) Γ is C^4 smooth curve,
- ($\Gamma 2$) Γ has no “near self-intersections”, i.e. there exists its strip neighborhood of a finite thickness which does not intersect with itself,
- ($\Gamma 3$) Γ is asymptotically straight in the sense that $\lim_{|s| \rightarrow \infty} \gamma(s) = 0$ and
- ($\Gamma 4$) Γ is not a straight line.

The operator, we are interested in, acts as a free Laplacian outside of the interaction support

$$(H_\beta \psi)(x) = -(\Delta \psi)(x)$$

for $x \in \mathbb{R}^2 \setminus \Gamma$ with the domain which can be written as $\mathcal{D}(H_\beta) = \{\psi \in H^2(\mathbb{R}^2 \setminus \Gamma) \mid \partial_{n_\Gamma} \psi(x) = \partial_{-n_\Gamma} \psi(x) = \psi'(x)|_\Gamma, -\beta \psi'(x)|_\Gamma = \psi(x)|_{\partial_+ \Gamma} - \psi(x)|_{\partial_- \Gamma}\}$. The vector n_Γ denotes the normal to Γ and $\psi(x)|_{\partial_\pm \Gamma}$ are the appropriate traces of the function ψ . For the purpose of the proofs we introduce curvilinear coordinates (s, u) along the curve in the same way as done in [6], i.e.

$$(x, y) = (\Gamma_1(s) + u\Gamma_2'(s), \Gamma_2(s) - u\Gamma_1'(s)). \quad (1)$$

As a result of the conditions ($\Gamma 1$) and ($\Gamma 2$) it can be shown that the map (1) is injective for all u small enough. We denote d as a maximum for which the map (1) is injective. A strip neighborhood around Γ of thickness $a < d$ is denoted by $\Omega_a := \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) < a\}$.

The quadratic form associated with the operator H_β was derived in [5] and it can be written as

$$h_\beta[\psi] = \|\nabla \psi\|^2 - \beta^{-1} \int_{\mathbb{R}} |\psi(s, 0_+) - \psi(s, 0_-)|^2 ds.$$

where we used the curvilinear coordinates in the strip neighborhood of the curve Γ for the functions $\psi \in C(\mathbb{R}^2) \cap H^1(\mathbb{R}^2 \setminus \Gamma)$ as $\psi(s, u)$. We also need to introduce the operator defined on the line as

$$S = -\frac{\partial^2}{\partial s^2} - \frac{1}{4}\gamma(s)^2, \quad (2)$$

with the domain $\mathcal{D}(S) = H^2(\mathbb{R})$. The eigenvalues of the operator S are denoted by μ_j with the multiplicity taken into account. Now we are ready to write down the main results of our paper.

Theorem 2.1. *Let an infinite curve Γ satisfy conditions ($\Gamma 1$)–($\Gamma 3$), then $\sigma_{\text{ess}}(H_\beta) \subseteq [\epsilon(\beta), \infty)$, where $\epsilon(\beta) \rightarrow -\frac{4}{\beta^2}$ holds as $\beta \rightarrow 0_+$.*

Theorem 2.2. *Let an infinite curve Γ satisfy assumptions $(\Gamma 1)$ – $(\Gamma 4)$, then H_β has at least one isolated eigenvalue below the threshold of the essential spectrum for all sufficiently small $\beta > 0$, and the j -th eigenvalue behaves in the strong coupling limit $\beta \rightarrow 0_+$ as*

$$\lambda_j = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(-\beta \ln(\beta)).$$

3. Bracketing estimates

For the proofs of both theorems we will need estimates of our operator H_β via Dirichlet and Neumann bracketing as done in [7]. We introduce the operators with added either Dirichlet or Neumann boundary conditions at the boundary of the strip neighborhood Ω_a of Γ . We introduce quadratic forms h_β^+ and h_β^- on the strip neighborhood of Γ which can be written as

$$h_\beta^\pm[\psi] = \|\nabla\psi\|^2 - \beta^{-1} \int_{\mathbb{R}} |\psi(s, 0_+) - \psi(s, 0_-)|^2 ds$$

with the domains $\mathcal{D}(h_\beta^+) = \tilde{H}_0^1(\Omega_a \setminus \Gamma)$ and $\mathcal{D}(h_\beta^-) = \tilde{H}^1(\Omega_a \setminus \Gamma)$. The operators associated with the quadratic forms h_β^\pm are denoted by H_β^\pm , respectively. With the help of Dirichlet-Neumann bracketing we are able to write the following inequality

$$-\Delta_{\mathbb{R}^2 \setminus \Omega_a}^N \oplus H_\beta^- \leq H_\beta \leq -\Delta_{\mathbb{R}^2 \setminus \Omega_a}^D \oplus H_\beta^+, \quad (3)$$

where $-\Delta_{\mathbb{R}^2 \setminus \Omega_a}^{N,D}$ denotes either Neumann or Dirichlet Laplacian on $\mathbb{R}^2 \setminus \Omega_a$ respectively. Neumann Laplacian and Dirichlet Laplacian are positive and as a result all the information about the negative spectrum, which we are interested in, is encoded in the operators H_β^\pm .

Now we rewrite the quadratic forms h_β^\pm in the curvilinear coordinates (1). We obtain expression which are analogical to those obtained in [6], i.e.

Lemma 3.1. *Quadratic forms h_β^+ , h_β^- are unitarily equivalent to quadratic forms q_β^+ and q_β^- which can be written as*

$$q^+[f] = \left\| \frac{\partial_s f}{g} \right\|^2 + \|\partial_u f\|^2 + (f, Vf) - \beta^{-1} \int_{\mathbb{R}} |f(s, 0_+) - f(s, 0_-)|^2 ds \\ + \frac{1}{2} \int_{\mathbb{R}} \gamma(s) (|f(s, 0_+)|^2 - |f(s, 0_-)|^2) ds$$

$$q^-[g] = q_D[g] - \int_{\mathbb{R}} \frac{\gamma(s)}{2(1+a\gamma(s))} |f(s, a)|^2 ds + \int_{\mathbb{R}} \frac{\gamma(s)}{2(1-a\gamma(s))} |f(s, -a)|^2 ds$$

defined on $\tilde{H}_0^1(\mathbb{R} \times ((-a, 0) \cup (0, a)))$ and $\tilde{H}^1(\mathbb{R} \times ((-a, 0) \cup (0, a)))$, respectively. The geometrically induced potential in these formulæ is given by

$$V(s, u) = \frac{u\gamma''}{2g^3} - \frac{5(u\gamma')^2}{4g^4} - \frac{\gamma^2}{4g^2}$$

with $g(s, u) := 1 + u\gamma(s)$.

The proof of this lemma can be done step by step as done in [6] so we omit the details.

We will also need cruder estimates by quadratic forms $b_\beta^\pm[f]$ which satisfy $b_\beta^-[f] \leq q^-[f] \leq h_\beta[f] \leq q^+[f] \leq b_\beta^+[f]$. The quadratic form $b_\beta^+[f]$ can be written as

$$b_\beta^+[f] = \|\partial_u f\|^2 + (1 - a\gamma_+)^{-2} \|\partial_s f\|^2 + (f, V^{(+)} f) - \beta^{-1} \int_{\mathbb{R}} |f(s, 0_+) - f(s, 0_-)|^2 ds + \frac{1}{2} \int_{\mathbb{R}} \gamma(s) (|f(s, 0_+)|^2 - |f(s, 0_-)|^2) ds$$

where $V^{(+)} := \frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{\gamma^2}{4(1+a\gamma_+)^2}$ and $f_+ := \max_{s \in \mathbb{R}} |f|$ denotes maximum of $|f|$. The quadratic form $b_\beta^-[f]$ can be written as

$$b_\beta^-[f] = \|\partial_u f\|^2 + (1 + a\gamma_+)^{-2} \|\partial_s f\|^2 + (f, V^{(-)} f) - \beta^{-1} \int_{\mathbb{R}} |f(s, 0_+) - f(s, 0_-)|^2 ds - \frac{1}{2} \int_{\mathbb{R}} \gamma(s) (|f(s, 0_+)|^2 - |f(s, 0_-)|^2) ds - \gamma_+ \int_{\mathbb{R}} |f(s, a)|^2 ds - \gamma_+ \int_{\mathbb{R}} |f(s, -a)|^2 ds$$

where $V^{(-)} = -\frac{a(\gamma'')_+}{2(1-a\gamma_+)^3} - \frac{5(a(\gamma')_+)^2}{4(1-a\gamma_+)^4} - \frac{\gamma^2}{4(1-a\gamma_+)^2}$. The operators B_β^\pm associated with $b_\beta^\pm[f]$ can be written as $B_\beta^\pm = U_a^\pm \otimes I + \int_{\mathbb{R}}^\oplus T_{a,\beta}^\pm(s) ds$ where U_a^\pm corresponds to the longitudinal variable s and $T_{a,\beta}^\pm(s)$ corresponds to the transversal variable u . The operators $T_{a,\beta}^\pm(s)$ act as $T_{a,\beta}^\pm(s)f = -f''$ with the domains

$$\begin{aligned} \mathcal{D}(T_{a,\beta}^+(s)) &= \{f \in H^2((-a, a) \setminus \{0\}) \mid f(a) = f(-a) = 0, \\ &\quad f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \frac{1}{2}\gamma(s)(f(0_+) + f(0_-))\} \\ \mathcal{D}(T_{a,\beta}^-(s)) &= \{f \in H^2((-a, a) \setminus \{0\}) \mid \mp\gamma_+ f(\pm a) = f'(\pm a), \\ &\quad f'(0_-) = f'(0_+) = -\beta^{-1}(f(0_+) - f(0_-)) + \frac{1}{2}\gamma(s)(f(0_+) + f(0_-))\}. \end{aligned}$$

The operators U_a^\pm act as $U_a^\pm f = -(1 \mp a\gamma_+)^{-2} f'' + V^{(\pm)} f$ with the domain $\mathcal{D}(U_a^\pm) = H^2(\mathbb{R})$. The operators $T_{a,\beta}^\pm(s)$ depend on the variable s , however, their negative spectrum is independent of s . Now we state two lemmata estimating the eigenvalues of operators $T_{a,\beta}^\pm(s)$ and U_a^\pm . Their proofs can be found in [6] so we omit the details.

Lemma 3.2. *Each of the operators $T_{a,\beta}^\pm(s)$ has exactly one negative eigenvalue $t_\pm(a, \beta)$, respectively, which is independent of s provided that $\frac{a}{\beta} > 2$ and $\frac{2}{\beta} > \gamma_+$. For all $\beta > 0$ sufficiently small these eigenvalues satisfy the inequalities*

$$-\frac{4}{\beta^2} - \frac{16}{\beta^2} \exp\left(-\frac{4a}{\beta}\right) \leq t_-(d, \beta) \leq -\frac{4}{\beta^2} \leq t_+(d, \beta) \leq -\frac{4}{\beta^2} + \frac{16}{\beta^2} \exp\left(-\frac{4a}{\beta}\right).$$

Lemma 3.3. *There is a positive C independent of a and j such that*

$$|\mu_j^\pm(a) - \mu_j| \leq C a j^2$$

holds for $j \in \mathbb{N}$ and $0 < a < \frac{1}{2\gamma_+}$, where $\mu_j^\pm(a)$ are the eigenvalues of U_a^\pm , respectively, with the multiplicity taken into account.

Now we are ready to prove our main theorems.

4. Proof of Theorem 2.1

First we prove the trivial case for the straight line. By separation of variables the spectrum is $\sigma(H_\beta) = \sigma_{\text{ess}}(H_\beta) = [-\frac{4}{\beta^2}, \infty)$.

The case for non-straight curve is done similarly as for the singular interaction supported by nonplanar surfaces in [8, 9]. The inclusion $\sigma_{\text{ess}}(H_\beta) \subseteq [\epsilon(\beta), \infty)$ can be rewritten as

$$\inf \sigma_{\text{ess}}(H_\beta) \geq \epsilon(\beta).$$

The inequality $H_\beta \geq H_\beta^- \oplus -\Delta_{\mathbb{R}^2 \setminus \Omega_d}^N$ implies that it is sufficient to check $\inf \sigma_{\text{ess}}(H_\beta^-) \geq \epsilon(\beta)$ in $L^2(\Omega_a)$ for $a < d$ because the operator $-\Delta_{\mathbb{R}^3 \setminus \Omega_d}^N$ is positive. Next we divide the curve Γ into two parts. First part is defined as $\Gamma_\tau^{\text{int}} := \{\Gamma(s) | s < \tau\}$ and the second one $\Gamma_\tau^{\text{ext}} := \Gamma \setminus \overline{\Gamma_\tau^{\text{int}}}$. The corresponding strip neighborhoods are defined as $\Omega_a^{\text{int}} := \{x(s, u) \in \Omega_a | s < \tau\}$ and $\Omega_a^{\text{ext}} := \{x(s, u) \in \Omega_a | s > \tau\}$. We introduce Neumann decoupled operators on $\Omega_a^{\text{int,ext}}$ as

$$H_{\beta,\tau}^{-,\text{int}} \oplus H_{\beta,\tau}^{-,\text{ext}}.$$

The operators $H_{\beta,\tau}^{-,\omega}$, $\omega = \text{int, ext}$ are associated with quadratic forms $h_{\beta,\tau}^{-,\omega}$ which can be written as

$$\begin{aligned} h_{\beta,\tau}^{-,\omega} = & \left\| \frac{\partial_s f}{g} \right\|^2 + \|\partial_u f\|^2 + (f, Vf) - \beta^{-1} \int_{\Gamma_\tau^\omega} |f(s, 0_+) - f(s, 0_-)|^2 ds \\ & + \frac{1}{2} \int_{\Gamma_\tau^\omega} \gamma(s) (|f(s, 0_+)|^2 - |f(s, 0_-)|^2) ds \\ & - \int_{\Gamma_\tau^\omega} \frac{\gamma(s)}{2(1 + a\gamma(s))} |f(s, a)|^2 ds + \int_{\Gamma_\tau^\omega} \frac{\gamma(s)}{2(1 - a\gamma(s))} |f(s, -a)|^2 ds \end{aligned}$$

with the domains $\tilde{H}^1(\Omega_a^\omega)$. Neumann bracketing implies that $H_{\beta,\tau}^- \geq H_{\beta,\tau}^{-,\text{int}} \oplus H_{\beta,\tau}^{-,\text{ext}}$. The spectrum of the operator $H_{\beta,\tau}^{-,\text{int}}$ is purely discrete [10] and as a result min-max principle implies that

$$\inf \sigma_{\text{ess}}(H_{\beta,\tau}^-) \geq \inf \sigma_{\text{ess}}(H_{\beta,\tau}^{-,\text{ext}}).$$

We denote the following expression $V_\tau := \inf_{|s| > \tau, u \in (-a, a)} V(s, u)$. The assumption (G2) gives us that

$$\lim_{\tau \rightarrow \infty} V_\tau = 0$$

With the help of Lemma 3.2 we can write the following estimates

$$\begin{aligned}
h_{\beta,\tau}^{-,\text{ext}}[f] &\geq \|\partial_u f\|^2 + V_\tau \|f\|^2 - \beta^{-1} \int_{\Gamma_\tau^{\text{ext}}} |f(s, 0_+) - f(s, 0_-)|^2 ds \\
&\quad + \frac{1}{2} \int_{\Gamma_\tau^{\text{ext}}} \gamma(s) (|f(s, 0_+)|^2 - |f(s, 0_-)|^2) ds \\
&\quad - \int_{\Gamma_\tau^{\text{ext}}} \frac{\gamma(s)}{2(1 + a\gamma(s))} |f(s, a)|^2 ds + \int_{\Gamma_\tau^{\text{ext}}} \frac{\gamma(s)}{2(1 - a\gamma(s))} |f(s, -a)|^2 ds \\
&\geq \left(V_\tau - \frac{4}{\beta^2} - \frac{16}{\beta^2} \exp\left(-\frac{4a}{\beta}\right) \right) \|f\|^2
\end{aligned}$$

Because we can choose τ arbitrarily large we obtain the the desired result.

5. Proof of Theorem 2.2

For the proof of the second theorem we use the inequalities (3) and Lemmata 3.2 and 3.3. First we put $a(\beta) = -\frac{3}{4}\beta \ln \beta$. Now with the explicit form of B_β^\pm in mind and the fact that $T_{a,\beta}^\pm(s)$ have exactly one negative eigenvalue we have that the spectra of B_β^\pm can be written as $t_\pm(d(\beta), \beta) + \mu_j^\pm(d(\beta))$. Using Lemmata 3.2 and 3.3 we obtain

$$t_\pm(a(\beta), \beta) + \mu_j^\pm(a(\beta)) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}(\beta |\ln \beta|).$$

The min-max principle along with the inequality (3) completes the proof.

Acknowledgments

The research was supported by the Czech Science Foundation within the project 14-06818S and by by Grant Agency of the Czech Technical University in Prague, grant No. SGS13/217/OHK4/3T/14.

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